BOUNDS FOR CERTAIN SUMS; A REMARK ON A CONJECTURE OF MAHLER

BY

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1. **Introduction.** The main part of this paper consists in a proof of the following

THEOREM 1. *Let*

$$Q(x, y) = Q_n(y)x^n + Q_{n-1}(y)x^{n-1} + \cdots + Q_0(y)$$

be a polynomial in x, y *with integral coefficients. Write* m *for the degree of* $Q_n(y)$, d *for the total degree of* $Q(x, y)$, *and let* $H \geqq 1$, $\rho > 1/3$ *be real numbers. Assume*

$$n \geqq 1, \ 3m \geqq n + 3; \qquad n\rho \geqq 1, \ m\rho \geqq 1, \tag{1}$$

and assume

$$Q(x, y) - k \tag{2}$$

has no rational linear factor if $k \neq 0$. *Then*

$$\sum_{|x| \leq H, |y| \leq H; Q(x,y) \neq 0} | Q(x, y) |^{-\rho} \leqq \gamma_{d\rho} H^{2/3}. \tag{3}$$

The sum is taken over integers x, y, *and the constant* $\gamma_{d\rho}$ *does not depend on the coefficients of* $Q(x, y)$, *except on the degree* d.

REMARK. Examples of the type $Q(x, y) = (x^2 - y)G(x, y) - 1$ show that the exponent 2/3 in (3) cannot be replaced by a constant less than 1/2.

By $P = P(x) = a_n x^n + \cdots + a_0$, we denote polynomials in x of degree n and with integral coefficients. Define $H(P)$ by

$$H(P) = \max(| a_n |, \cdots, | a_0 |),$$

and write $D(P)$ for the discriminant of P.

THEOREM 2. *Let* $\rho > 1/3$ *be real and assume*

$$n \geqq 3, \qquad (n - 1)\rho \geqq 1. \tag{4}$$

Then

$$\sum_{P; H(P) \leq H; D(P) \neq 0} | D(P) |^{-\rho} \leqq \delta_{n\rho} H^{n-1/3}. \tag{5}$$

Presented to the Society, January 26, 1961; received by the editors February 21, 1961.

REMARKS. Only the bound H^{n+1} is trivial; the bound H^n was given in [3]. Our result is probably far from best possible. For $n=2$, the bound H follows easily from the methods of [2](1).

In Mahler's classification of transcendental numbers ζ, numbers $\theta_n(\zeta)$ ($n=1, 2, \cdots$) are defined as follows: $\theta_n(\zeta)$ is the least upper bound of the set of real numbers σ such that there exists a sequence of polynomials $P_1(x), P_2(x), \cdots$ of degree n and different from each other, satisfying

$$|P_i(\zeta)| < H(P)^{-n\sigma} \qquad (i = 1, 2, \cdots).$$

Mahler [6] conjectured $\theta_n(\zeta)=1$ almost everywhere. It is known that $\theta_n(\zeta) \geqq 1$ a.e. (see, for instance, [7, p. 69]). $\theta_1(\zeta)=1$ a.e. follows from a theorem of Khintchine [4], $\theta_2(\zeta)=1$ a.e. was proved in [5; 1; 2; 9], $\theta_3(\zeta)=1$ a.e. has recently been proved by Volkmann(2). The best estimate so far for arbitrary n was $\theta_n(\zeta) \leqq 2-2/n$ a.e. ($n \geqq 2$), and had been obtained by Kasch-Volkmann [3].

THEOREM 3. *Assume $n \geqq 3$ and suppose τ is a real number such that*

$$\sum_{P;H(P)\leqq H;D(P)\neq 0} |D(P)|^{-1/2} = O(H^\tau). \tag{6}$$

Then

$$\theta_n(\zeta) \leqq 1 + \frac{\tau - 2}{n} \quad \textit{almost everywhere.}$$

Combining Theorems 2, 3 we obtain

THEOREM 4.

$$\theta_n(\zeta) \leqq 2 - 7/3n \qquad \textit{a.e.}\ (n \geqq 3).$$

2. **Two lemmas.** We first give a short proof of the following result of Kasch-Volkmann [3].

LEMMA 1. *To any $n \geqq 1$ and any $\rho > 0$, there exists a constant γ such that*

$$\sum_{|x|\leqq H;\, P(x)\neq 0} |P(x)|^{-\rho} \leqq \begin{cases} \gamma H^{1-n} & \text{if } n\rho < 1, \\ \gamma \log H & \text{if } n\rho = 1, \\ \gamma & \text{if } n\rho > 1 \end{cases} \tag{7}$$

for any P.

Proof. Write $P(x) = a_n(x-\alpha_1) \cdots (x-\alpha_n)$ and put $\alpha_i' = R\alpha_i$ ($i=1, \cdots, n$) and

$$\langle x \rangle = \min_{i=1,\ldots,n} |x - \alpha_i'|.$$

(1) *Added in proof*: H. Davenport in a paper to appear in Mathematika gives the bound $O(H^2)$ if $n=3$, $\rho=1/2$.

(2) According to a letter to the author.

Since an inequality of the type $k \leqq \langle x \rangle < k+1$ has at most $2n$ integral solutions x, since there are at most $2H+1$ integers x in $|x| \leqq H$, and since the function $g(k) = k^{-n\rho}$ is decreasing, we obtain

$$\sum_{|x| \leqq H;\, \langle x \rangle \geqq 1} \langle x \rangle^{-n\rho} \leqq 2n \sum_{k=1}^{2H+1} k^{-n\rho}. \tag{8}$$

There are at most $2n$ integers x having $\langle x \rangle < 1$, hence their contribution to the sum (7) is at most $2n$. A bound for the remaining sum is furnished by $|P(x)| \geqq \langle x \rangle^n$ and (8), whence the result.

LEMMA 2. *Assume the integers n, m and the real $\rho > 1/3$ satisfy* (1). *Define a sequence $\nu_0, \nu_1, \cdots$ by*

$$\nu_0 = 2/3 - (4m\rho)^{-1}, \tag{i}$$

$$\nu_i = mn^{-1}\nu_{i-1} + (2n\rho - 1)(3n\rho)^{-1} \qquad (i = 1, 2, \cdots). \tag{ii}$$

Then the elements of the sequence are positive and increasing. The sequence either tends to infinity or it has a finite limit larger than 1.

Proof. We have $\nu_0 > 0$ and

$$\begin{aligned} \nu_1 &= mn^{-1}(2/3 - (4m\rho)^{-1}) + (2n\rho - 1)(3n\rho)^{-1} \\ &= 2/3 + (2m\rho - 3/4 - 1)(3n\rho)^{-1} \\ &> 2/3 > \nu_0. \end{aligned}$$

The sequence is increasing because $\nu_{j-1} < \nu_j$ together with (ii) implies $\nu_j < \nu_{j+1}$.

If $m \geqq n$, then the sequence tends to infinity and the lemma is true. Assume therefore $m < n$. Then there exists some ν such that

$$mn^{-1}\nu + (2n\rho - 1)(3n\rho)^{-1} = \nu. \tag{9}$$

Clearly, $mn^{-1}K + (2n\rho - 1)(3n\rho)^{-1}K < K$ if $\nu < K$. Using this property of ν as well as the fact that the sequence is increasing, we see that the sequence is bounded by ν and hence has a limit. Because of (9), the limit equals ν. Solving (9) we find

$$\nu = (2n\rho - 1)(3n\rho - 3m\rho)^{-1},$$

and $3m > n + \rho^{-1}$, $3m\rho > n\rho + 1$, $2n\rho - 1 > 3n\rho - 3m\rho$ gives $\nu > 1$.

Let t be the smallest integer with $\nu_t \geqq 1$. Putting

$$\mu_i = \nu_i - 2/3 \qquad (i = 1, 2, \cdots), \tag{10}$$

we obtain

$$\begin{aligned} &0 < \mu_1 < \cdots < \mu_{t-1} < 2/3 \leqq \mu_{t'}, \\ &\mu_i = mn^{-1}\nu_{i-1} - (3n\rho)^{-1} \qquad (i = 1, 2, \cdots), \\ &n\rho\mu_i - m\rho\nu_{i-1} = -1/3 \qquad (i = 1, 2, \cdots). \end{aligned} \tag{11}$$

Write

(12) $$Q_n(y) = l(y - \beta_1) \cdots (y - \beta_m),$$

(13) $$Q(x, y) = Q_n(y)(x - \alpha_1(y)) \cdots (x - \alpha_n(y)) \qquad \text{if } Q_n(y) \neq 0,$$

and put $\beta_i' = R\beta_i$, $\alpha_j'(y) = R\alpha_j(y)$ and

(14) $$\{y\} = \min_{i=1,\ldots,m} |y - \beta_i'|,$$

(15) $$\{x \mid y\} = \min_{j=1,\ldots,n} |x - \alpha_j'(y)|.$$

We split the sum (3) into $t+2$ parts $\sum_{00}$, $\sum_0$, $\sum_1, \cdots, \sum_t$, where

$\sum_{00}$ consists of the terms of the sum where $\{y\} = 0$,

$\sum_{0}$ consists of the terms with $0 < \{y\} \leqq H^{\nu_0}$,

$\sum_{j}$ $(j = 1, \cdots, t-1)$ consists of the terms with $H^{\nu_{j-1}} < \{y\} \leqq H^{\nu_j}$,

$\sum_{t}$ consists of terms satisfying $H^{\nu_{t-1}} < \{y\}$.

Each of the sums $\sum_j$ $(j=1, \cdots, t)$ will be split into three parts $\sum_{j1}$, $\sum_{j2}$, $\sum_{j3}$ where the pairs (x, y) involved satisfy

$$\sum_{j1}: \qquad \{x \mid y\} \geqq 1$$

$$\sum_{j2}: \qquad H^{-\mu_j} \leqq \{x \mid y\} < 1$$

$$\sum_{j3}: \qquad \{x \mid y\} < H^{-\mu_j}.$$

3. Bounds for $\sum_{00}$, $\sum_0$, $\sum_{j1}$, $\sum_{j2}$.

Lemma 3. $\sum_{00} = O(H^{2/3})$.

More explicitly, $\sum_{00} \leqq \gamma H^{2/3}$, where γ depends on d and ρ only. In all the equations of this section and the next, the O-symbol is to be understood in this way.

Proof. $\{y\} = 0$ is equivalent with $Q_n(y) = 0$. There are at most m integers y with $Q_n(y) = 0$. For given y_0 having $Q_n(y_0) = 0$, there are two alternatives. Either $Q_{n-1}(y_0) = \cdots = Q_1(y_0) = 0$, or there exists some $h \geqq 1$ such that $Q_h(y_0) \neq 0$.

In the first case we have $Q_0(y_0) = 0$ because otherwise the polynomial (2) with $k = Q_0(y_0) \neq 0$ would have the real linear factor $y - y_0$. Hence we have $Q(x, y_0) \equiv 0$ identically in x and there is no contribution to the sum (3) with $y = y_0$.

In the second case $Q(x, y_0)$ is a polynomial in x of some degree between 1 and d, and Lemma 1 yields

$$\sum_{|x|\leq H;\,Q(x,y_0)\neq 0} |Q(x, y_0)|^{-\rho} = O(H^{1-\rho}) = O(H^{2/3}).$$

LEMMA 4. $\sum_0 = O(H^{2/3})$.

Proof. For fixed y_0 with $\{y_0\} > 0$, $Q(x, y_0)$ is a polynomial in x of degree n, and $n\rho \geqq 1$ together with Lemma 1 gives the bound $O(\log H)$ for the sum over x. There are at most $(2H^{\nu_0}+1)m$ integers y_0 with $0 < \{y_0\} \leqq H^{\nu_0}$, and Lemma 4 follows from $H^{\nu_0} \log H = O(H^{2/3})$.

LEMMA 5. $\sum_{j1} = O(H^{2/3})$ $\quad (j = 1, \cdots, t)$.

Proof. We have $|Q(x, y)| \geqq \{y\}^m\{x|y\}^n$. Just as in (8), one can see that

$$\sum_{|y|\leq H;\,\{y\}\geq 1} \{y\}^{-m\rho} \leqq 2m \sum_{k=1}^{2H+1} k^{-m\rho} = O(\log H)$$

and, for fixed y,

$$\sum_{|x|\leq H;\,\{x|y\}\geq 1} \{x|y\}^{-n\rho} \leqq 2n \sum_{k=1}^{2H+1} k^{-n\rho} = O(\log H).$$

The lemma follows.

LEMMA 6. $\sum_{j2} = O(H^{2/3})$ $\quad (j = 1, \cdots, t)$.

Proof. This time we have $|Q(x, y)| \geqq \{y\}^m\{x|y\}^n \geqq H^{m\nu_{j-1}}H^{-n\mu_j}$. There are at most $2n$ integers x with $\{x|y\} < 1$ for given y. Therefore

$$\begin{aligned}\sum_{j2} &\leqq 2n(2H+1)H^{-m\rho\nu_{j-1}+n\rho\mu_j}\\ &= O(H^{2/3})\end{aligned}$$

according to (11).

4. **Bounds for $\sum_{j3}$.** The domain $H^{\nu_{j-1}} < \{\eta\} \leqq H^{\nu_j}$ $(j = 1, \cdots, t-1)$ or the domain $H^{\nu_{t-1}} < \{\eta\}$ consists of at most $2m$ strips parallel to the x-axis. The intersection of these strips with $|\xi| \leqq H$, $|\eta| \leqq H$ consists of at most $2m$ rectangles. The length of such a rectangle in the direction of the x-axis is $2H$, the length in the direction of the y-axis at most $2mH^{\nu_j}$ $(j = 1, \cdots, t-1)$ or $2mH$. From now on, we keep j fixed, and R will be a fixed rectangle of the type described above. We shall give bounds for the terms of $\sum_{j3}$ where $(x, y) \in R$.

Write

$$\alpha_{ij}(y) = (\alpha_i(y) + \alpha_j(y))/2 \qquad (1 \leqq i, j \leqq n;\ i \neq j),$$

where $\alpha_j(y)$ is defined in (13). The elementary symmetric polynomials of the $C_{n,2}$ functions $\alpha_{ij}(y)$ are polynomials in y of degree $O(1)$, and therefore there exists a polynomial $R(x, y)$ of degree $C_{n,2}$ in x and of total degree $O(1)$ having

$R(\alpha_{ij}(y), y) = 0$ for $1 \leqq i, j \leqq n, i \neq j$ and y arbitrary. Put $S(x, y) = Q(x, y)R(x, y)$. Writing $\alpha_j'(y)$ for the real part of $\alpha_j(y)$ we find

$$S(\alpha_j'(y), y) = 0 \qquad (j = 1, \cdots, n).$$

The real solutions of $S(\xi, \eta) = 0$ will form certain curves in the plane. Their intersection with R^* where R^* is the rectangle containing R in which the condition $|\xi| \leqq H$ is replaced by $|\xi| \leqq H+1$ will consist of a number of curves of the type

$$x = x(y), \qquad \phi \leqq y \leqq \psi, \tag{16}$$

where $x'(y)$, $x''(y)$ exist and either

$$x''(y) \geqq 0 \tag{16a}$$

or

$$x''(y) \leqq 0, \tag{16b}$$

and perhaps some line-segments of the type

$$y = c, \qquad -H - 1 \leqq x \leqq H + 1. \tag{17}$$

We denote the curves by $C_1, \cdots, C_q$ and have $q = O(1)$.

By $N(C_l)$ $(l = 1, \cdots, q)$ denote the set of integral pairs $(x, y) \in R$ such that for suitable ζ

$$(\zeta, y) \in C_l, \qquad |x - \zeta| < H^{-\mu_i}.$$

$\{x|y\} < H^{-\mu_i}$ implies $|x - \alpha_k'(y)| < H^{-\mu_i}$ for some k ($q \leqq k \leqq n$). Hence $(x, y) \in R$ together with $\{x|y\} < H^{-\mu_i}$ implies $(x, y) \in H(C_l)$ for some $l(1 \leqq l \leqq q)$, and we have

$$\begin{aligned} \sum_{j3} &\leqq \sum_{l=1}^{q} \sum_{(x,y)\in N(C_l)} |Q(x, y)|^{-\rho} \\ &= \sum_{l=1}^{q} A(C_l). \end{aligned} \tag{18}$$

LEMMA 7. *$A(C_l) = O(H^{2/3})$ if C_l is of the type* (17).

Proof. For fixed $y = c$ with $\{c\} > 0$, $Q(x, c)$ is a polynomial in x of degree n, and $n\rho \geqq 1$ together with Lemma 1 gives the bound $O(\log H)$.

From now on we shall assume $C = C_l$ is of the type (16a). There are trivial changes in the argument if C is of the type (16b). Construct the convex hull of $N(C)$ and in this convex hull consider the lattice-points (x, y) such that $(x - \epsilon, y)$ is not in the hull if $\epsilon > 0$. This set of lattice-points will be written $S(C)$, *spine* of C. $S(C)$ is not necessarily contained in $N(C)$.

LEMMA 8. *The number of points of $N(C)$ which are not in $S(C)$ is $O(H^{2/3})$.*

Proof. We may assume H is so large that $2H^{-\mu_j}<1$. The points of $S(C)$, let us say $(x_1, y_1), \cdots, (x_g, y_g)$, can be ordered such that $y_1<y_2<\cdots<y_g$. Any $(x, y)\in N(C)$ has $y_1\leqq y\leqq y_g$. We shall prove that the number of points of $N(C)$ not in $S(C)$ with $y_i\leqq y\leqq y_{i+1}$ is at most

$$2(y_{i+1}-y_i)H^{-\mu_j}.$$

Since

$$\sum_{i=1}^{g-1}(y_{i+1}-y_i)\leqq 2mH^{\nu_j}$$

and since $\nu_j-\mu_j=2/3$ according to (10), the lemma follows.

Write $x=x_i(y)$ $(i=1, \cdots, g-1)$ for the equation of the line through (x_i, y_i) and (x_{i+1}, y_{i+1}). Obviously, $x_i=x_i(y_i)$, $x_{i+1}=x_i(y_{i+1})$. We have $x>x_i(y)$ for every $(x, y)\in N(C)$ which is not in $S(C)$. As before, $x=x(y)$ is the equation of C. We find

$$\text{(19)}\qquad \begin{aligned} x_i &> x(y_i)-H^{-\mu_j} && (i=1,\cdots,g),\\ x(y_i)-x_i(y_i) &< H_-^{\mu_j} && (i=1,\cdots,g-1), \end{aligned}$$

and similarly

$$\text{(20)}\qquad x(y_{i+1})-x_i(y_{i+1})<H^{-\mu_j}\qquad (i=1,\cdots,g-1).$$

Using (19), (20) and $(x(y)-x_i(y))''\geqq 0$ we find

$$x(y)-x_i(y)<H^{-\mu_j}\text{ if } y_i\leqq y\leqq y_{i+1}\qquad (i=1,\cdots,g-1).$$

Any $(x, y)\in N(C)$ with $y_i\leqq y\leqq y_{i+1}$ has therefore

$$x<x(y)+H^{-\mu_j}<x_i(y)+2H^{-\mu_j}.$$

Thus we have to show that there are at most $2(y_{i+1}-y_i)H^{-\mu_j}$ lattice-points in the parallelogram $y_i\leqq y\leqq y_{i+1}$, $x_i(y)<x<x_i(y)+2H^{-\mu_j}$, or at most $2bH^{-\mu_j}$ lattice-points in the parallelogram

$$\text{(21)}\qquad 0\leqq y\leqq b,\qquad \frac{a}{b}y<x<\frac{a}{b}y+2H^{-\mu_j},$$

where $a=x_{i+1}-x_i$, $b=y_{i+1}-y_i$. a and b are relatively prime. Writing (ζ) for the difference between the smallest integer not smaller than ζ and ζ itself, (21) can be rewritten

$$\text{(22)}\qquad 0\leqq y\leqq b,\qquad 0<\left(\frac{a}{b}\,y\right)<2H^{-\mu_j}.$$

But the number of integral solutions of (22) is equal to the largest integer not exceeding $2bH^{-\mu_j}$.

LEMMA 9. *There are in* $S(C)$ *at most*

$$O(\min (r^{1/3}H^{2/3}, H)$$

points having $0<|Q(x, y)|\leqq r$.

Proof. We may assume $2dr<H$, because otherwise $\min(r^{1/3}H^{2/3}, H)$ gives the trivial estimate $O(H)$. There are at most d collinear points with $Q(x, y)=k$, $k\neq 0$. Hence there are at most $2dr$ collinear points having $0<|Q(x, y)|\leqq r$.

Write $(x^{(1)}, y^{(1)}), \cdots, (x^{(p)}, y^{(p)})$ for the points of $S(C)$ with $0<|Q(x, y)|\leqq r$, and assume $y^{(1)}< \cdots <y^{(p)}$. Introduce the vectors

$$v_i = (x^{(i+1)} - x^{(i)}, y^{(i+1)} - y^{(i)}) \qquad (i = 1, \cdots, p-1).$$

Since the points of $S(C)$ are on a convex polygon, it follows that $v_i=v_{i+k}$ implies $v_i = v_{i+1} = \cdots = v_{i+k}$ and that the points $(x^{(i)}, y^{(i)}), \cdots, ((x^{(i+k+1)}, y^{(i+k+1)})$ are collinear. Hence at most $2dr$ of the vectors v_i can be equal. We have

$$\sum_{i=1}^{p-1} |v_i| \leqq 6H + 4, \tag{23}$$

where $|v|$ denotes the length of v.

Order the set of all the nonzero vectors w of R^2 with integral components in such a way that

$$1 = |w_1| \leqq |w_2| \leqq \cdots.$$

Apparently $|w_i|>\gamma_1 i^{1/2}(\gamma_1>0)$, and therefore

$$|w_1| + \cdots + |w_i| > \gamma_2 i^{3/2}.$$

If $p\leqq 2dr$, then $p<(2dr)^{1/3}H^{2/3}$ and the lemma is true. Hence we may assume $p>2dr$. Write $p=2drs+u$, where $0\leqq u<2dr$.

$$\begin{aligned}|v_1| + \cdots + |v_p| &\geqq 2dr(|w_1| + \cdots + |w_s|)\\ &\geqq 2dr\gamma_2 s^{3/2} \geqq \gamma_3(d)rq^{3/2}r^{-3/2}\\ &= \gamma_3(d)p^{3/2}r^{-1/2}.\end{aligned}$$

Using (23) we obtain $p^{3/2}r^{-1/2} = O(H)$, $p = O(r^{1/3}H^{2/3})$.

LEMMA 10.

$$\sum_{(x,y)\in S(C)} |Q(x, y)|^{-\rho} = O(H^{2/3}).$$

Proof. Write $a(r)$ for the number of points of $S(C)$ with $Q(x, y)=r$. We have to show that the (finite) sum

$$\sum_{r=-\infty}^{\infty}{}' a(r)|r|^{-\rho} = O(H^{2/3}).$$

(The prime indicates that the term $r=0$ is omitted.) Using partial summation we find for $N>H$

$$\begin{aligned}
\sum_{r=-N}^{N}{}' a(r)\,|r|^{-\rho} &= \sum_{r=1}^{N} (a(r) + a(-r))r^{-\rho} \\
&= \sum_{r=1}^{N}\Big(\sum_{k=-r}^{r}{}' a(k)\Big)(r^{-\rho} - (r+1)^{-\rho}) + \sum_{k=-N}^{N}{}' a(k)(N+1)^{-\rho} \\
&= O\Big(\sum_{r=1}^{H} r^{1/3}H^{2/3}r^{-1-\rho} + \sum_{r=H+1}^{N} Hr^{-1-\rho} + HN^{-\rho}\Big) \\
&= O(H^{2/3} + H^{1-\rho} + H^{1-\rho}) \\
&= O(H^{2/3}).
\end{aligned}$$

Proof of Theorem 1. As explained in §2, it is sufficient to give bounds for $\sum_{00}$, $\sum_{0}$, $\sum_{j1}$, $\sum_{jj'}$, $\sum_{j3}$ $(j=1, \cdots, t)$. Bounds for sums of the first four types are given in §3. To estimate $\sum_{j3}$, it is enough to estimate $A(C_l)$, as is seen by (18). This is done in Lemmas 7 through 10.

REMARK. The crucial lemma of the proof is Lemma 10. Theorem 1 could be improved if this lemma could be improved.

5. **Proof of Theorem** 2. The discriminant $D(P)$ of a polynomial P is a polynomial $D(a_0, a_1, \cdots, a_n)$ in the coefficients of P.

LEMMA 11.

(i) $D(a_0, \cdots, a_n) = \pm n^n a_0^{n-1}a_n^{n-1} + a_0^{n-2}R_{n-2} + \cdots + R_0$, *where* $R_{n-2}, \cdots, R_0$ *are polynomials in* $a_1, a_2, \cdots, a_n$.

(ii) *The total degree of* $D(a_0, \cdots, a_n)$ *in* a_0 *and* a_n *is* $2n-2$, *and the only term of this degree is* $\pm n^n a_0^{n-1}a_n^{n-1}$.

(iii) $D_n(a_0, \cdots, a_{n-1}, 0) = \pm a_{n-1}^2 D_{n-1}(a_0, \cdots, a_{n-1})$.

(iv) $D(a_0, \cdots, a_n) = D(a_n, \cdots, a_0)$.

Proof. Using $\pm a_n D(P) = R(P, P')$, where $R(P, P')$ is the resultant of P and P', and the determinant representation of $R(P, P')$ (see, for instance, [8, §§29–31]; observe that we write $P=a_nx^n+ \cdots +a_0$ while van der Waerden writes $P=a_0x^n+ \cdots +a_n$), we find

$$D(a_0, \cdots, a_n) = \pm \left|\begin{matrix}
1 & a_{n-1} & \cdots & a_1 & a_0 & & & \\
 & a_n & a_{n-1} & \cdots & a_1 & a_0 & & \\
 & & \cdots & & & & & \\
 & & & a_n & a_{n-1} & \cdots & a_1 & a_0 \\
n & (n-1)a_{n-1} & \cdots & a_1 & & & & \\
 & na_n & (n-1)a_{n-1} & \cdots & a_1 & & & \\
 & & \cdots & & & & & \\
 & & & na_n & (n-1)a_{n-1} & \cdots & a_1 &
\end{matrix}\right| \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n-1 \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \end{matrix}$$

(i) and (ii) follow immediately from this representation while (iii) follows after a short computation using the determinant representation of $D_n(a_0, \cdots, a_n)$ is well as of $D_{n-1}(a_0, \cdots, a_{n-1})$.

To prove (iv) we have to show that $D(P)=D(\overline{P})$ where $P=a_n(x-\alpha_1)\cdots(x-\alpha_n)$ and $\overline{P}=a_0(x-\alpha_1^{-1})\cdots(x-\alpha_n^{-1})$. Now

$$\begin{aligned} D(\overline{P}) &= a_0^{n-1}\prod_{i\neq j}(\alpha_i^{-1}-\alpha_j^{-1}) = \prod_{i\neq j}(\alpha_i-\alpha_j)a_0^{n-1}\prod_{i=1}^{n}\alpha_i^{-(n-1)} \\ &= \prod_{i\neq j}(\alpha_i-\alpha_j)a_0^{n-1}(a_n/a_0)^{n-1}(-1)^{n(n-1)} = a_n^{n-1}\prod_{i\neq j}(\alpha_i-\alpha_j) \\ &= D(P). \end{aligned}$$

Lemma 12. *Let $n\geqq 3$. Keep $a_1, \cdots, a_{n-1}$ fixed and write*

$$D(x, y) = D(x, a_1, \cdots, a_{n-1}, y).$$

Then

(a) $D(x, y)=\pm n^n x^{n-1}y^{n-1}+x^{n-2}Q_{n-2}(y)+\cdots+Q_0(y)$.

(b) *The total degree of $D(x, y)$ is $2n-2$ and only the term $\pm n^n x^{n-1}y^{n-1}$ has this degree.*

(c) *$D(x, y)-k$ has no linear factor if $k\neq 0$.*

Proof. (a) and (b) follow from (i) and (ii) of the previous lemma. As for (c), assume there would be some $k\neq 0$ such that $D(x, y)-k$ had a linear factor. If $x=\alpha z+\beta$, $y=\gamma z+\delta$ were the parameter equation of this line, then we had

$$f(z) = D(\alpha z+\beta, \gamma z+\delta) \equiv k \neq 0$$

identically in z.

If $\alpha\gamma\neq 0$, then $f(z)$ is a polynomial of degree $2n-2$ according to (b), a contradiction. Hence either $\alpha=0$ or $\gamma=0$, and because of (iv) we may assume $\gamma=0$. We have $D(z, \delta)\equiv k\neq 0$ identically in z, which is conceivable only with $\delta=0$, according to (a). Hence $D(z, 0)\equiv k\neq 0$. Using (iii) and applying (i) to D_{n-1} we find

$$\begin{aligned} 0\neq k \equiv D(z, 0) &= \pm\, a_{n-1}^2 D_{n-1}(z, a_1, \cdots, a_{n-1}) \\ &= \pm\, a_{n-1}^2(\pm(n-1)^{n-1} z^{n-2}\, a_{n-1}^{n-2} + S(z)), \end{aligned}$$

where $S(z)$ is of degree $\leqq n-3$. But $k\neq 0$ implies $a_{n-1}\neq 0$ and $D(z, 0)$ is therefore a polynomial of degree $n-2>0$, and we reach a contradiction.

Proof of Theorem 2. Lemma 12 enables us to apply Theorem 1 on $D(x, y)$ and we obtain

$$\begin{aligned} \sum_{P;H(P)\leq H;D(P)\neq 0} |D(P)|^{-\rho} &= \sum_{|a_1|\leq H}\cdots\sum_{|a_{n-1}|\leq H}\ \sum_{|x|\leq H,|y|\leq H;D(x,y)\neq 0} |D(x, y)|^{-\rho} \\ &= O(H^{n-1/3}). \end{aligned}$$

6. The conjecture of Mahler.

LEMMA 13. *Suppose σ is a number such that the sum*

$$\sum_{H=1}^{\infty}\left\{\sum_{P;H(P)=H;D(P)\neq 0} H^{-2-\sigma n}|D(P)|^{-1/2}\right\}$$

is convergent. Then $\theta_n(\zeta)\leqq 1+\sigma$ almost everywhere.

Proof. This lemma follows from the argument on pages 448–449 of [3].

Proof of Theorem 3. Assume that (6) holds for some τ and put $\sigma=\sigma_\epsilon=(\tau-2)n^{-1}+\epsilon$ for some $\epsilon>0$. Using partial summation we find

$$\sum_{H=1}^{N}\left\{\sum_{P;H(P)=H;D(P)\neq 0} H^{-2-\sigma n}|D(P)|^{-1/2}\right\}$$
$$=O\left[\sum_{H=1}^{N}\left\{\sum_{P;H(P)\leq H;D(P)\neq 0} H^{-3-\sigma n}|D(P)|^{-1/2}\right\} + N^{-2-\sigma n}\sum_{P;H(P)\leq N;D(P)\neq 0}|D(P)|^{-1/2}\right]$$
$$=O\left(\sum_{H=1}^{N} H^{-3-\sigma n+\tau}+N^{-2-\sigma n+\tau}\right)$$
$$=O(1).$$

Hence Lemma 13 yields $\theta_n(\zeta)\leqq 1+\sigma_\epsilon$ almost everywhere. Since $\epsilon>0$ was arbitrary, we obtain $\theta_n(\zeta)\leqq 1+(\tau-2)n^{-1}$ almost everywhere.

REFERENCES

1. J. W. S. Cassels, *Some metrical theorems in Diophantine approximation.* V, *On a conjecture of Mahler*, Proc. Cambridge Philos. Soc. vol. 47, (1951) pp. 18–21

2. F. Kasch, *Über eine metrische Eigenschaft der S-Zahlen*, Math. Z. vol. 70 (1958) pp. 263–270.

3. F. Kasch, and B. Volkmann, *Zur Mahlerschen Vermutung über S-Zahlen*, Math. Ann. vol. 136, (1958) pp. 442–453.

4. A. Y. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. vol. 92 (1924) pp. 115–125.

5. J. F. Kubilyus, *On the application of a method of Vinogradoff in a problem in metrical number-theory*, (Russian), Dokl. Akad. Nauk SSSR (N.S.) vol. 67 (1949) pp. 783–786.

6. K. Mahler, *Über das Mass der Menge aller S-Zahlen.* Math. Ann. vol. 106 (1932) pp. 131–139.

7. T. Schneider, *Einführung in die transzendenten Zahlen*, Berlin-Göttingen-Heidelberg, Springer, 1957.

8. B. L. van der Waerden, *Algebra*, erster Teil, 4th ed., Berlin-Göttingen-Heidelberg, Springer, 1955.

9. B. Volkmann, *Zur Mahlerschen Vermutung in Komplexen*, Math. Ann. vol. 140 (1960) pp. 351–359.

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